On the Integral Equations of Laser Theory*

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Summary—**Mathematical comments are made on a class of integral equations with complex symmetric kernels which occur in laser theory. It is pointed out that integral equations of this class do not have some of the well-known properties of integral equations with Hermitian kernels. In particular, the usual extremal principle by which the eigenvalues of a Hermitian kernel may be estimated using the Rayleigh-Ritz procedure does not apply to integral equations with complex symmetric kernels. It is suggested that the use of variational techniques to calculate diffraction losses in laser interferometers leads to results of doubtful accuracy.**

RECENTLY considerable interest has developed in certain integral equations which arise in laser theory. These equations are usually of the form

$$\int_a^b K(x, y)\psi(y)dy = \kappa\psi(x), \tag{1}$$

with a complex-valued symmetric kernel which is not Hermitian, that is

$$K(x, y) = K(y, x) \quad \text{but} \quad K(x, y) \neq \overline{K(y, x)}. \tag{2}$$

A typical such equation, which relates to a laser with parallel plane reflectors, is

$$\int_{-1}^{1} e^{ik(x-y)^2} f(y)dy = \kappa f(x), \tag{3}$$

where x is real.

It is the purpose of this paper to point out that integral equations with complex symmetric kernels do not, as a class, possess certain of the familiar properties of equations with Hermitian kernels. In particular, it is suggested that the use of variational techniques[1,2] to calculate approximate eigenvalues of an equation such as (3) leads to results of doubtful accuracy.

Most of the standard textbook treatment of homogeneous linear integral equations of the second kind relates to equations with Hermitian kernels, including real symmetric kernels as a special case. It is well known, for example, that 1) every Hermitian kernel which does not vanish almost everywhere has at least one eigenvalue (we do not count zero as an eigenvalue), 2) the eigenvalues of a Hermitian kernel are necessarily real, 3) the eigenvalues of a Hermitian kernel $H(x, y)$ are the stationary values of the variational quotient

$$Q[\phi] = \frac{\int_a^b \int_a^b \overline{\phi(x)} H(x, y)\phi(y)dxdy}{\int_a^b \overline{\phi(x)}\phi(x)dx}, \tag{4}$$

and 4) $Q[\phi]$ is a lower bound for the largest eigenvalue κ_1, with $\kappa_1 = \max Q[\phi]$.

It does not seem to be universally recognized that these properties of Hermitian kernels are not shared by complex symmetric kernels. Part of the theory does go through for normal kernels, that is, kernels for which

$$\int_a^b K(x, z)\overline{K(y, z)}dz = \int_a^b \overline{K(z, x)}K(z, y)dz; \tag{5}$$

but the kernel of (3) is not even of this class. In general, for complex symmetric kernels 1), 2), and 4) are not true, and 3) is true only in a modified form which is of limited usefulness. We shall discuss each of these points briefly.

To see that a complex symmetric kernel need not have eigenvalues, one has only to consider the integral equation

$$\int_{-1}^{1} (1 + i\sqrt{3}\,x)(1 + i\sqrt{3}\,y)\psi(y)dy = \kappa\psi(x). \tag{6}$$

If there were an eigenfunction, it would have to be of the form

$$\psi(x) = A(1 + i\sqrt{3}\,x). \tag{7}$$

But

$$\int_{-1}^{1} (1 + i\sqrt{3}\,y)^2 dy = 0, \tag{8}$$

and so the equation has no nontrivial solution. If the sign of one of the i's in (6) were changed, of course, the kernel would be Hermitian, and would have the eigenvalue 4.

The question whether a complex symmetric kernel has any eigenvalues must at present be settled on a more or less individual basis. So far as we know, no mathematical theorem has ever been published which would guarantee *a priori* that such an equation as (3) actually has eigenvalues. For this particular example, of course, the existence of eigenvalues has been made morally certain by the iterative numerical computations of Fox and Li.[3]

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[1] W. Culshaw, "Further considerations on Fabry-Perot type resonators," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-10, pp. 331–339; September, 1962.

[2] C. L. Tang, "On diffraction losses in laser interferometers," *Appl. Optics*, vol. 1, pp. 768–770; November, 1962.

[3] A. G. Fox and T. Li, "Resonant modes in a maser interferometer," *Bell Sys. Tech. J.*, vol. 40, pp. 453–488; March, 1961.

If a complex symmetric kernel has eigenvalues, they are likely to be complex. This is certainly the case for (3), as well as for the simpler example which will be treated below.

It has been correctly pointed out[1,2] that the eigenvalues of a complex symmetric kernel are given by the stationary values of the ratio

$$R[\phi] = \frac{\int_a^b \int_a^b \phi(x)K(x, y)\phi(y)dxdy}{\int_a^b \phi^2(x)dx}. \tag{9}$$

This differs from the expression $Q[\phi]$ which can be used in the Hermitian case, in that $R[\phi]$ does not involve any complex conjugates. Formally it is true that $R[\phi]$ is a variational expression for the eigenvalues; that is, if $\phi(x)$ is "close" to some eigenfunction $\psi(x)$, then $R[\phi]$ will in some sense be "very close" to the corresponding eigenvalue κ. However, the complex-valued ratio $R[\phi]$ is not as useful for complex symmetric kernels as the real quotient $Q[\phi]$ is for Hermitian kernels, because $R[\phi]$ does not lead to an extremal principle. In particular it is *not* true that $|R[\phi]|$ is a lower bound on $|\kappa_1|$. The following example shows that $|R[\phi]|$ is not generally a bound of any kind.

Consider the integral equation

$$\int_{-1}^{1} [1 + ik\sqrt{3}\,(x + y)]\psi(y)dy = \kappa\psi(x), \tag{10}$$

where k is real and greater than zero. It is easy to show that the eigenvalues of (10) are

$$\kappa_{1,2} = 1 \pm (1 - 4k^2)^{1/2}, \tag{11}$$

corresponding to the eigenfunctions

$$\psi_{1,2}(x) = 1 \pm (1 - 4k^2)^{1/2} + 2ik\sqrt{3}\,x, \tag{12}$$

where the upper sign goes with the first subscript and the lower sign with the second subscript. The variational ratio for (10) is

$$R[\phi] = \frac{\int_{-1}^{1}\int_{-1}^{1} \phi(x)[1 + ik\sqrt{3}\,(x + y)]\phi(y)dxdy}{\int_{-1}^{1} \phi^2(x)dx}. \tag{13}$$

If the difference between $\phi(x)$ and $\psi_1(x)$ or $\psi_2(x)$ is a small quantity of order ϵ, then $R[\phi]$ differs from κ_1 or κ_2 by a complex quantity of order ϵ^2, but in general nothing can be said about the *phase* of the error. If the trial function $\phi(x)$ is unrestricted, it is perfectly possible for $R[\phi]$ to be zero or to be infinite. For example, straightforward calculation yields

$$R[1 - 3x^2] = 0, \qquad R[1 + i\sqrt{3}\,x] = \infty, \tag{14}$$

except that the second expression takes the form 0/0 in the degenerate case $k = \frac{1}{2}$.

The upshot of all this is that one is always free to guess an approximate eigenfunction $\phi(x)$ of a complex symmetric kernel and to call $R[\phi]$ an approximate eigenvalue, but there is no way of knowing *a priori* how close the result is to the actual eigenvalue κ, or in what direction $R[\phi]$ differs from κ. Furthermore it appears impossible to use the Rayleigh-Ritz procedure, as has been suggested,[1] to refine a complex eigenvalue, since in the absence of a maximum or minimum principle what criterion can be used to adjust the parameters of a trial function to get closer to the exact eigenvalue?

We shall now comment briefly on the numerical results which have been obtained[1,2] by applying the variational technique to an equation equivalent to (3). In dimensionless form, the equation which has been studied may be written[4]

$$N^{1/2}e^{i\pi/4}\int_{-1}^{1} e^{-i\pi N(x-y)^2}\psi(y)dy = \kappa\psi(x). \tag{15}$$

where $N(=a^2/b\lambda)$ is a parameter depending upon the laser dimensions and the wavelength. The first two eigenvalues κ_1 and κ_2 have been estimated for $N = \frac{1}{2}$, 1, 2, 3, 4, by setting

$$\phi_1(x) = \cos \pi x/2,$$
$$\phi_2(x) = \sin \pi x, \tag{16}$$

in the variational quotient (9).

The magnitudes of both $|\kappa_1|$ and $|\kappa_2|$ approach unity as N increases, and these magnitudes[1,2] agree well with those which Fox and Li obtained by a lengthy iterative calculation. However, the quantity of greatest physical interest is not κ but the relative power loss per transit, which is $1 - |\kappa|^2$. Tables I and II show the power loss as computed for the lowest even and odd modes by the variational procedure[1,2] and by the iterative technique.[5]

Inspection of Tables I and II indicates that agreement between the two calculations of loss is not very good, and that it gets worse as N increases. It has been asserted[1] that the variational method becomes more accurate the larger the value of N. If one is interested in power loss, there is good reason to believe that the reverse is true, since the closer $|\kappa|$ is to unity, the more seriously do small errors of unknown sense affect the value of $1 - |\kappa|^2$. Also, since one does not know that $|R[\phi]| \leq |\kappa_1|$ for a complex symmetric kernel, there is

[4] The eigenvalue κ was denoted by $1/\gamma$ in Tang[2] and Fox and Li.[3]

[5] The column headed "Fox and Li" contains the numerical values used to plot Fox and Li's Fig. 8.[3] These values were kindly supplied by the authors.

TABLE I

POWER LOSS PER TRANSIT FOR DOMINANT MODE BETWEEN PARALLEL STRIP MIRRORS

N	Loss $=1-\|\kappa_1\|^2$	
	Variational method	Iterative method
$\frac{1}{2}$	0.1639	0.1740
1	0.0607	0.0800
2	0.0217	0.0320
3	0.0108	0.0194
4	0.0080	0.0129

TABLE II

POWER LOSS PER TRANSIT FOR LOWEST ODD MODE BETWEEN PARALLEL STRIP MIRRORS

N	Loss $=1-\|\kappa_2\|^2$	
	Variational method	Iterative method
$\frac{1}{2}$	0.6035	0.5397
1	0.2401	0.2698
2	0.0863	0.1207
3	0.0468	0.0703
4	0.0308	0.0478

no reason for presuming that the variational values of $|\kappa_1|$ are better than the iterative ones because they are larger than the latter values. In this connection it is interesting to note that the variational values for $|\kappa_2|$ do not all lie on the same side of the iterative results.

The foregoing remarks are not intended to keep anyone from applying variational principles to integral equations with complex symmetric kernels if he so desires. One pays for the "efficiency" of the variational technique by not knowing how accurate the approximate eigenvalue really is, or whether, if one tries another assumed eigenfunction, the second approximation is better or worse than the first. We merely wish to point out how little is actually known, in a mathematical sense, about integral equations of this type. We believe that computer-generated numerical solutions, although admittedly laborious, will furnish the most reliable results that we are likely to get from the integral equations of laser theory, at least until someone develops an adequate analytic theory of integral equations with complex symmetric kernels.

The Dipolar Resonance of the Cylindrical Low-Pressure Arc Discharge*

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Summary—When an em wave of fixed frequency is incident on the cylindrical positive column of a low-pressure arc discharge, nearly complete absorption occurs at a definite value of discharge current I_0 as the discharge current is varied. I_0 yields a plasma electron density which corresponds to the well-known cylindrical, or dipolar, plasma resonance frequency f_0. The ratio f_p/f_0 where f_p is the ordinary (plane) plasma frequency, has been determined by others using a quasi-static approach. In this paper a dynamic approach is used, and comparison is made with the quasi-static approach. Agreement is within 3 per cent for values of $\beta_0 a$ less than 0.25. For $\beta_0 a$ equal to 0.60, the discrepancy in the quasi-static method is 15 per cent.

Theoretical calculations as well as experimental evidence indicate that the electron sheath, which exists on the outside surface of the positive column, plays a significant role in the location of the dipolar plasma resonance.

Application of the results of this paper improve the agreement between theory and experiment for the Plasma Microwave Coupler described by Steier and Kaufman.

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I. INTRODUCTION

WHEN AN EM WAVE of fixed frequency is incident on the cylindrical positive column of a low-pressure mercury-vapor discharge, a spectrum of resonances of reflection and transmission occurs as the discharge current of the positive column is varied. Nearly complete absorption occurs at a definite value of current I_0 which yields a plasma density corresponding to the well-known cylindrical, or dipolar, plasma resonance frequency. If the discharge current is maintained constant, an analogous spectrum of resonances occurs as the frequency of the incident em wave is varied. Nearly complete absorption occurs at a definite value of frequency f_0 which is the dipolar plasma resonance frequency. If the plasma electron density is assumed to be linearly related to the discharge current,